4.2 Energy and Power

Definition 4.12. For a signal $g(t)$, the instantaneous power $p(t)$ dissipated in the 1-Ω resister is $p_g(t) = |g(t)|^2$ regardless of whether $g(t)$ represents a voltage or a current. To emphasize the fact that this power is based upon unity resistance, it is often referred to as the **normalized (instantaneous) power**.

Definition 4.13. The total (normalized) ***energy*** of a signal $g(t)$ is given by

energy within ±T

$$E_g = \int_{-\infty}^{+\infty} p_g(t)dt = \int_{-\infty}^{+\infty} |g(t)|^2 dt = \lim_{T\to\infty} \int_{-T}^{T} |g(t)|^2 dt.$$

4.14. By the **Parseval's theorem** discussed in 2.43, we have

$$E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(f)|^2 df.$$

Definition 4.15. The average (normalized) ***power*** of a signal $g(t)$ is given by

$$P_g = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |g(t)|^2 dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |g(t)|^2 dt.$$

energy per unit time within ±T

Definition 4.16. To simplify the notation, there are two operators that used angle brackets to define two frequently-used integrals:

"mean"

(a) The "**time-average**" operator:

$$\langle g \rangle \equiv \langle g(t) \rangle \equiv \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} g(t)dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} g(t)dt \tag{42}$$

(b) The **inner-product** operator:

$$\langle g_1, g_2 \rangle \equiv \langle g_1(t), g_2(t) \rangle = \int_{-\infty}^{\infty} g_1(t) g_2^*(t) dt \tag{43}$$

4.17. Using the above definition, we may write

- $E_g = \langle g, g \rangle = \langle G, G \rangle$ where $G = \mathcal{F}\{g\}$
- $P_g = \left\langle |g|^2 \right\rangle$

Important properties:

1. $\langle c \rangle = c$

$\langle 5 \rangle = 5$

2. $\langle a g_1(t) + b g_2(t) \rangle$

$= a \langle g_1(t) \rangle + b \langle g_2(t) \rangle$

- Parseval's theorem: $\langle g_1, g_2 \rangle = \langle G_1, G_2 \rangle$
  where $G_1 = \mathcal{F}\{g_1\}$ and $G_2 = \mathcal{F}\{g_2\}$

4.18. Time-Averaging over Periodic Signal: For ***periodic*** signal $g(t)$ with period T_0, the time-average operation in (42) can be simplified to

$$\langle g \rangle = \frac{1}{T_0} \int_{T_0} g(t)dt$$

where the integration is performed over a period of g.

Example 4.19. $\langle \cos(2\pi f_0 t + \theta) \rangle = \begin{cases} 0, & f_0 \neq 0, \\ \cos\theta, & f_0 = 0 \end{cases}$

Similarly, $\langle \sin(2\pi f_0 t + \theta) \rangle = \begin{cases} 0, & f_0 \neq 0, \\ \sin\theta & f_0 = 0. \end{cases}$

Example 4.20. $\left\langle \cos^2(2\pi f_0 t + \theta) \right\rangle = \left\langle \frac{1}{2} + \frac{1}{2}\cos(2\pi(2f_0)t + 2\theta) \right\rangle$

$\cos^2 x = \frac{1}{2}(1 + \cos(2x))$

$= \begin{cases} 1/2, & f_0 \neq 0, \\ \cos^2\theta, & f_0 = 0, \end{cases}$

Example 4.21. $\left\langle e^{j(2\pi f_0 t + \theta)} \right\rangle = \langle \cos(2\pi f_0 t + \theta) + j\sin(2\pi f_0 t + \theta) \rangle$

$= \begin{cases} 0, & f_0 \neq 0, \\ e^{j\theta}, & f_0 = 0, \end{cases}$

Example 4.22. Suppose $g(t) = ce^{j2\pi f_0 t}$ for some (possibly complex-valued) constant c and (real-valued) frequency f_0. Find P_g.

$P_g = \langle |g|^2 \rangle = \langle |ce^{j2\pi f_0 t}|^2 \rangle = \langle |c|^2 \underbrace{|e^{j2\pi f_0 t}|^2}_{1} \rangle = |c|^2$

4.23. When the signal $g(t)$ can be expressed in the form $g(t) = \sum_k c_k e^{j2\pi f_k t}$ and the f_k are distinct, then its (average) power can be calculated from

$$P_g = \sum_k |c_k|^2$$

Example 4.24. Suppose $g(t) = 2e^{j6\pi t} + 3e^{j8\pi t}$. Find P_g.

$c_1 = 2$ $\quad$ $c_2 = 3$

$f_1 = 3$ $\quad$ $f_2 = 4$

$P_g = |c_1|^2 + |c_2|^2 = 2^2 + 3^2 = 4 + 9 = 13$

Example 4.25. Suppose $g(t) = 2e^{j6\pi t} + 3e^{j6\pi t}$. Find P_g.

$g(t) = 5e^{j2\pi(3)t}$

$P_g = |5|^2 = 25$

Example 4.26. Suppose $g(t) = \cos(2\pi f_0 t + \theta)$. Find P_g.

Here, there are several ways to calculate P_g. We can simply use Example 4.20. Alternatively, we can first decompose the cosine into complex exponential functions using the Euler's formula:

$\cos(2\pi f_0 t + \theta)$

$f_0 \neq 0$: $\underbrace{\frac{1}{2} e^{j2\pi f_0 t} e^{j\theta}}_{c_1} + \underbrace{\frac{1}{2} e^{-j2\pi f_0 t} e^{-j\theta}}_{c_2} \Rightarrow P = |c_1|^2 + |c_2|^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2}$

$f_0 = 0$: $\cos\theta \Rightarrow P = \langle \cos^2\theta \rangle = \cos^2\theta$

4.27. The (average) power of a sinusoidal signal $g(t) = A\cos(2\pi f_0 t + \theta)$ is

$$P_g = \begin{cases} \frac{1}{2}|A|^2, & f_0 \neq 0, \\ |A|^2 \cos^2\theta, & f_0 = 0. \end{cases}$$

This property means any sinusoid with nonzero frequency can be written in the form

$$g(t) = \sqrt{2P_g}\cos(2\pi f_0 t + \theta).$$

4.28. Extension of 4.27: Consider sinusoids $A_k \cos(2\pi f_k t + \theta_k)$ whose frequencies are positive and distinct. The (average) power of their sum

$$g(t) = \sum_k A_k \cos(2\pi f_k t + \theta_k)$$

is

$$P_g = \frac{1}{2}\sum_k |A_k|^2.$$

$f_1 > 0 \quad \neq \quad f_2 > 0$

Example 4.29. Suppose $g(t) = 2\cos\left(2\pi\sqrt{3}t\right) + 4\cos\left(2\pi\sqrt{5}t\right)$. Find P_g.

$A_1 \qquad A_2$

$$P_g = \frac{|A_1|^2}{2} + \frac{|A_2|^2}{2} = \frac{1}{2}\left(2^2 + 4^2\right) = \frac{1}{2}(4 + 16) = 10$$

Example 4.30. Suppose $g(t) = 3\cos(2t) + 4\cos(2t - 30^\circ) + 5\sin(3t)$. Find P_g.

$$3\angle 0^\circ + 4\angle -30^\circ$$

$$\approx 6.77\angle -17.2^\circ$$

$$\Updownarrow$$

$$6.77\cos(2t - 17.2^\circ)$$

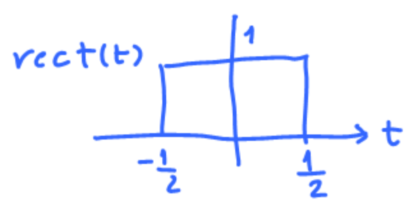
$$P_g \approx \frac{1}{2}\left(6.77^2 + 5^2\right) \approx 35.4$$

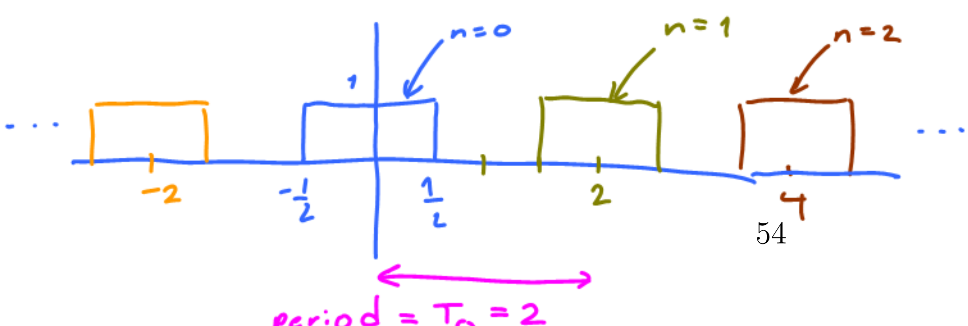
4.31. For ***periodic*** signal $g(t)$ with period T_0, there is also no need to carry out the limiting operation to find its (average) power P_g. We only need to find an average carried out over a single period:

$$P_g = \frac{1}{T_0}\int_{T_0} |g(t)|^2 dt.$$

Example 4.32.

$$g(t) = \sum_{n=-\infty}^{\infty} \mathrm{rect}(t - 2n)$$





$$P_g = \frac{1}{T_0}\int_{T_0} |g^2(t)|\,dt$$

$= \frac{1}{2}$ → finite, ≠0 ⇒ g is a power signal

$E_g = \infty$ ⇒ g is not an energy signal

4.33. When the Fourier series expansion (to be reviewed in Section 4.3) of the signal is available, it is easy to calculate its power:

(a) When the corresponding Fourier series expansion $g(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t}$ is known,

$$P_g = \sum_{k=-\infty}^{\infty} |c_k|^2.$$

(b) When the signal $g(t)$ is real-valued and its (compact) trigonometric Fourier series expansion $g(t) = c_0 + 2\sum_{k=1}^{\infty} |c_k| \cos(2\pi k f_0 t + \angle\phi_k)$ is known,

$$P_g = c_0^2 + 2\sum_{k=1}^{\infty} |c_k|^2.$$

Definition 4.34. Based on Definitions 4.13 and 4.15, we can define three distinct classes of signals:

(a) If E_g is finite and nonzero, g is referred to as an ***energy signal***.

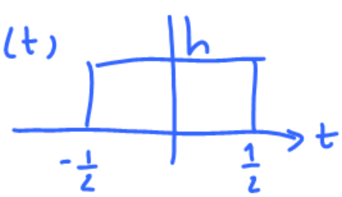
(b) If P_g is finite and nonzero, g is referred to as a ***power signal***.

(c) Some signals[17] are neither energy nor power signals.

- Note that the power signal has infinite energy and an energy signal has zero average power; thus the two categories are disjoint.

Example 4.35. Rectangular pulse

$g(t) = h \times rect(t)$



finite

$$E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt = h^2$$

$\neq 0$

g is an energy signal.

$$P_g = \langle |g(t)|^2 \rangle \equiv \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |g(t)|^2 dt = 0$$

$\Rightarrow$ g is not a power signal

[17]Consider $g(t) = t^{-1/4} 1_{[t_0,\infty)}(t)$, with $t_0 > 0$.

Example 4.36. Sinc pulse

Example 4.37. For $\alpha > 0$, $g(t) = Ae^{-\alpha t}1_{[0,\infty)}(t)$ is an energy signal with $E_g = |A|^2/2\alpha$.

Example 4.38. The rotating phasor signal $g(t) = ce^{j(2\pi f_0 t+\theta)}$ is a power signal with $P_g = |c|^2$.

Example 4.39. The sinusoidal signal $g(t) = A\cos(2\pi f_0 t + \theta)$ is a power signal with $P_g = |A|^2/2$.

4.40. Consider the transmitted signal

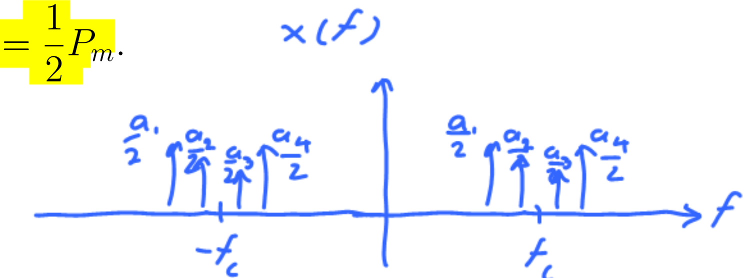
$$x(t) = m(t)\cos(2\pi f_c t + \theta)$$

in DSB-SC modulation. Suppose $M(f - f_c)$ and $M(f + f_c)$ do not overlap (in the frequency domain).

(a) If $m(t)$ is a power signal with power P_m, then the average transmitted power is

$$P_x = \frac{1}{2}P_m.$$

"proof"



$P_m = |a_1|^2 + |a_2|^2 + \cdots + |a_4|^2$

$P_x = \left(\left|\frac{a_1}{2}\right|^2 + \left|\frac{a_2}{2}\right|^2 + \cdots \left|\frac{a_4}{2}\right|^2 \right) \times 2 = \frac{1}{2}P_m$

- Q: Why is the power (or energy) reduced?

- Remark: When $x(t) = \sqrt{2}m(t)\cos(2\pi f_c t + \theta)$ (with no overlapping between $M(f - f_c)$ and $M(f + f_c)$), we have $P_x = P_m$.

(b) If $m(t)$ is an energy signal with energy E_m, then the transmitted energy is

$$E_x = \frac{1}{2}E_m.$$

Example 4.41. Suppose $m(t) = \cos(2\pi f_c t)$. Find the average power in $x(t) = m(t)\cos(2\pi f_c t)$.